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Cyclic Just-In-Time Sequences Are Optimal

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Abstract. Consider the *Product Rate Variation* problem. Given n products $1, \ldots, i, \ldots, n$, and n positive integer demands $d_1, \ldots, d_i, \ldots, d_n$. Find a sequence $\alpha = \alpha_1, \ldots, \alpha_T$, $T = \sum_{i=1}^{n} d_i$, of the products, where product i occurs exactly d_i times that always keeps the *actual* production level, equal the number of product i occurrences in the prefix $\alpha_1, \ldots, \alpha_t$, $t = 1, \ldots, T$, and the *desired* production level, equal $r_i t$, where $r_i = d_i/T$, of each product i as close to each other as possible. The problem is one of the most fundamental problems in sequencing flexible just-in-time production systems. We show that if β is an optimal sequence for $d_1, \ldots, d_i, \ldots, d_n$, then concatenation β^m of m copies of β is an optimal sequence for $md_1, \ldots, md_i, \ldots, md_n$.

Key words: Optimization, assignment problem, apportionment problem, convex functions, just-in-time systems

1. Introduction

We study the following optimization problem. Given n products $1, \ldots, i, \ldots, n$, n positive integers (*demands*) $d_1, \ldots, d_i, \ldots, d_n$, and n convex and symmetric functions $f_1, \ldots, f_i, \ldots, f_n$ of a single variable, all assuming minimum 0 at 0. Find a sequence $\alpha = \alpha_1, \ldots, \alpha_T$, $T = \sum_{i=1}^{n} d_i$, of products $1, \ldots, i, \ldots, n$, where product i occurs exactly d_i times that minimizes

$$F(\alpha) = \sum_{i=1}^{n} \sum_{t=1}^{T} f_i(x(\alpha)_{it} - r_i t),$$

where $x(\alpha)_{it}$ = the number of product i occurrences (*copies*) in the prefix $\alpha_1, \ldots, \alpha_t$, $t = 1, \ldots, T$, and $r_i = d_i/T$, $i = 1, \ldots, n$.

This problem is known as the *Product Rate Variation (PRV) problem* in the literature, see Kubiak [7], Bautista, Companys and Corominas [3, 4], and Balinski and Shahidi [2]. The problem is fundamental in flexible just-in-time production systems, where sequences, we refer to them as *JIT* sequences, that make these systems tick must keep the actual, equal $x(\alpha)_{it}$, and the desired, equal $r_i t$, production levels of each product i as close to each other as possible all the time, Monden [14], Miltenburg [13], Groenevelt [6], and Vollman et al [17].

This paper focuses on the question whether there always exists an optimal *cyclic* solution to the PRV problem. This question can formally be stated as follows:

Let β be an optimal sequence for $d_1, \dots, d_i, \dots, d_n$. Is β^m, for any $m \geqslant 1$, an optimal sequence for $md_1, \dots, md_i, \dots, md_n$, where β^m is a concatenation of m copies of β?

An affirmative answer to the question will add an important theoretical justification to the usual for just-in-time systems practice of repeating relatively short sequence to build a sequence for a longer time horizon, Monden [14] and Miltenburg [13]. Also, either answer to the question has obvious consequences for the computational time complexity of any algorithm for JIT sequences, see Monden [14], Miltenburg [13], Kubiak and Sethi [9, 10], and Steiner and Yeomans [15] for available algorithms.

The question has recently received growing attention. Bautista, Companys and Corominas [4] have proven an affirmative answer provided that $f_i = f$ for all i, and function f is convex and symmetric with minimum $f(0) = 0$. The cornerstone of their proof is an observation that even with the constraints $x(\alpha)_{iT} = d_i$, $i = 1, \dots, n$, relaxed there still exists an optimal sequence α^* such that $x(\alpha^*)_{iT} = d_i$ for all i. However, Kubiak and Kovalyov [8] have shown that this observation no longer holds when f_i's are not identical, but still convex and symmetric with minimum 0 at 0.

Kubiak and Kovalyov [8] have shown that if all f_i functions are convex, symmetric and equal in the interval $(0, 1)$, then again the answer is affirmative. However, they give an example for which the answer is negative if at least one f_i function is asymmetric.

All the affirmative answers obtained thus far rely on two crucial observations. First, if $\alpha = \beta\gamma$ where β and γ are sequences for $ad_1, \dots, ad_n$ and $bd_1, \dots, bd_n$ respectively, with a and b being positive integers, then $F(\alpha) = F(\beta) + F(\gamma)$, see Miltenburg [13]. Second, even if one relaxes the constraints $x(\alpha)_{iT} = d_i$, $i = 1, \dots, n$, there will still exist an optimal sequence α^* such that $x(\alpha^*)_{iT} = d_i$ for all i. The latter relies on a simple exchange of two copies of different products in a given sequence that does not increase the value of the sequence. However, since the copies exchanged are of different products this technique may increase the value of the sequence in general when f_i's are different. Thus the simple exchange method will fail to work in a general case unless some exchanges are forbidden. Consequently, a more sophisticated exchange method will be developed in this paper to prove optimality of cyclic solutions. This method will limit exchanges to copies of products that occupy positions at the same distance from the *ends* 1 and T of a sequence, we assume for the time being that all demands are even. That is the exchange will only be allowed between positions 1 and T, 2 and $T - 1$, 3 and $T - 2$ etc. We show that any such an exchange does not increase the value of the sequence, and that it is possible to carry out the exchanges so that the resulting sequence has the number of copies of each product *equally* split between its two halves. The *existence* of this desired distribution of product copies will be assured by the *Hall's theorem*, see for example [12] for this theorem. Our method will rely on the idea of ideal positions of copies of a product and the assignment problem equivalent to the PRV problem which is based on this idea, both were introduced by Kubiak and Sethi [9, 10].

Steiner and Yeomans [16] have proven an affirmative answer for the *min-max* problem with $f_i(x) = |x|$.

Balinski and Shahidi [2] have approached PRV from a different angle, an *axiomatic* one. They have proposed a rule, called $\phi^{1/2}$, and used it to recursively build a complete sequence. The rule has been selected from a potentially infinite number of rules so that the sequences it builds have required properties (axioms), one of which is that the complete sequence is cyclic. This elegant approach to the PRV problem follows the well-known axiomatic approach to the *apportionment* problem, Balinski and Young [1]. The axiomatic approach to the PRV problem was suggested by Bautista, Companys and Corominas [3] who were first to notice a link between the PRV problem and the apportionment problem.

The paper is organized as follows. Section 2 presents an assignment problem equivalent to the PRV problem, and it is entirely based on Kubiak and Sethi [9, 10]. Section 3 proves special properties of the costs in this assignment problem for *even* instances, that is the ones with all demands being even. Section 4 presents a three-step transformation that splits all copies of each product *equally* between the two halves of a feasible sequence for an even instance, and at the same time does not increase the value of solution. This construction relies on the properties of the assignment problem discussed in Sections 2 and 3. Section 5 proves that cyclic JIT sequences are optimal. Finally, Section 6 presents conclusions and open questions.

2. Ideal Positions and Reduction to Assignment Problem

Let $X = \{(i, j, l) | i = 1, \dots, n;\ j = 1, \dots, d_i;\ l = 1, \dots, T\}$. Following Kubiak and Sethi [9, 10], define cost $C^i_{jt} \geqslant 0$ for $(i, j, t) \in X$ as follows:

$$C^i_{jt} = \begin{cases} \sum_{l=t}^{Z^i_j - 1} \psi^i_{jl}, & \text{if } t < Z^i_j, \\ 0, & \text{if } t = Z^i_j, \\ \sum_{l=Z^i_j}^{t-1} \psi^i_{jl}, & \text{if } t > Z^i_j, \end{cases} \tag{1}$$

where for symmetric functions f_i, $Z^i_j = \lceil (2j-1)/(2r_i) \rceil$ is called the *ideal position* for the j-th copy of product i, and

$$\begin{aligned} \psi^i_{jl} &= |f_i(j - lr_i) - f_i(j - 1 - lr_i)| = \\ &= \begin{cases} f_i(j - lr_i) - f_i(j - 1 - lr_i), & \text{if } l < Z^i_j, \\ f_i(j - 1 - lr_i) - f_i(j - lr_i), & \text{if } l \geqslant Z^i_j. \end{cases} \end{aligned} \tag{2}$$

Let $S \subseteq X$, we define $V(S) = \sum_{(i,j,l) \in S} C^i_{jl}$, and call S feasible if it satisfies the following three constraints:

(A) For each $l, l = 1, \dots, T$, there is exactly one pair (i, j), $i = 1, \dots, n$; $j = 1, \dots, d_i$ such that $(i, j, l) \in S$.
(B) For each pair (i, j), $i = 1, \dots, n$; $j = 1, \dots, d_i$, there is exactly one $l, l = 1, \dots, T$, such that $(i, j, l) \in S$.
(C) If $(i, j, l), (i, j', l') \in S$ and $l < l'$, then $j < j'$.

Constraints (A) and (B) are the well known assignment problem constraints, constraints (C) impose an order on copies of a product and will be elaborated upon later.

Consider any set S of T triples (i, j, l) satisfying (A), (B), and (C). Let $\alpha(S) = \alpha(S)_1, \dots, \alpha(S)_T$, where $\alpha(S)_l = i$ if $(i, j, l) \in S$ for some j, be a sequence corresponding to S. By (A) and (B) sequence $\alpha(S)$ is feasible for $d_1, \dots, d_n$. The following theorem ties $F(\alpha(S))$ and $V(S)$ for any feasible S.

THEOREM 1. *We have*

$$F(\alpha(S)) = V(S) + \sum_{i=1}^{n} \sum_{t=1}^{T} \inf_{j} f_i(j - tr_i). \tag{3}$$

Proof. See Kubiak and Sethi [10]. □

Unfortunately, an optimal set S can not be found by simply solving the assignment problem with constraints (A) and (B), and the costs as in (1), for which many efficient algorithms exist, see for example Kuhn [11]. The reason is constraint (C), which is not of assignment type. Informally, (C) ties up copy j of a product with the j-th ideal position for the product and it is necessary for Theorem 1 to hold. In other words, for a set S satisfying (A) and (B) but not (C) we may have inequality in (3).

THEOREM 2. *If S satisfies (A) and (B), then S' satisfying (A), (B) and (C), and such that*

$$V(S) \geqslant V(S'),$$

can be constructed in $O(T)$ steps. Furthermore, each product occupies the same positions in $\alpha(S')$ as it does in $\alpha(S)$.

Proof. See Kubiak and Sethi [10]. □

3. Properties of the C^i_{jt}'s for Even Instances

In this section we study only the even instances of the PRV problem. That is we assume that demands are of the form $2d_1, \dots, 2d_n$ for some positive integers $d_1, \dots, d_n$, and feasible sequences have length $2T$, where $T = \sum_{i=1}^{n} d_i$. We prove important properties of the costs C^i_{jt} in the assignment problem introduced in Section 2.

LEMMA 1. *Let* $1 \leqslant t < Z_j^i < t' \leqslant 2T$ *for some* $i = 1, \dots, n$ *and* $j = 1, \dots, 2d_i$. *If* $t + t' > 2Z_j^i$, *then* $C_{jt'}^i \geqslant C_{jt}^i$. *If* $t + t' < 2Z_j^i$, *then* $C_{jt}^i \geqslant C_{jt'}^i$.

Proof. By definitions (1) and (2) we have

$$C_{jt'}^i = \sum_{l=Z_j^i}^{t'-1} \psi_{jl}^i = \sum_{l=Z_j^i}^{t'-1} [f_i(j-1-lr_i) - f_i(j-lr_i)]$$

and

$$C_{jt}^i = \sum_{l=t}^{Z_j^i-1} \psi_{jl}^i = \sum_{l=t}^{Z_j^i-1} [f_i(j-lr_i) - f_i(j-1-lr_i)].$$

Consider $C_{jt'}^i - C_{jt}^i$ for $t + t' > 2Z_j^i$, and $C_{jt}^i - C_{jt'}^i$ for $t + t' < 2Z_j^i$. In the former case, by "matching" $l = Z_j^i + a$ from $C_{jt'}^i$ with $l = Z_j^i - a$ from C_{jt}^i, for $a = 1, \dots, Z_j^i - t$, and by using the convexity of f_i, we get

$$C_{jt'}^i - C_{jt}^i \geqslant 2 \sum_{a=1}^{Z_j^i-t} [f_i(j-1-Z_j^i r_i) - f_i(j - Z_j^i r_i)].$$

In the latter case, by "matching" $l = Z_j^i - b - 2$ from C_{jt}^i with $l = Z_j^i + b$ from $C_{jt'}^i$, for $b = 0, \dots, t' - 1 - Z_j^i$, and by using the convexity of f_i, we get

$$C_{jt}^i - C_{jt'}^i \geqslant 2 \sum_{b=0}^{t'-1-Z_j^i} [f_i(j - Z_j^i r_i + r_i) - f_i(j-1-Z_j^i r_i + r_i)].$$

By definition of ideal positions

$$Z_j^i = \lceil (2j-1)/(2r_i) \rceil.$$

Thus,

$$\begin{aligned} j - 1 - Z_j^i r_i &= -1/2 - \epsilon r_i, \\ j - Z_j^i r_i &= 1/2 - \epsilon r_i, \\ j - Z_j^i r_i + r_i &= 1/2 + (1-\epsilon) r_i, \end{aligned}$$

and

$$j - 1 - Z_j^i r_i + r_i = -1/2 + (1-\epsilon) r_i,$$

for some $0 \leqslant \epsilon < 1$. We have

$$|-1/2 - \epsilon r_i| = 1/2 + \epsilon r_i \geqslant |1/2 - \epsilon r_i|,$$

and

$$|1/2 + (1-\epsilon)r_i| = 1/2 + (1-\epsilon)r_i \geqslant |-1/2 + (1-\epsilon)r_i|.$$

Therefore,

$$f_i(j-1-Z_j^i r_i) \geqslant f_i(j-Z_j^i r_i)$$

and

$$f_i(j-Z_j^i r_i + r_i) \geqslant f_i(j-1-Z_j^i r_i + r_i),$$

consequently, $C_{jt'}^i - C_{jt}^i \geqslant 0$ for $t+t' > 2Z_j^i$, and $C_{jt}^i - C_{jt'}^i \geqslant 0$ for $t+t' < 2Z_j^i$ and the lemma holds. □

LEMMA 2. *We have*

$$C^i_{(2d_i+1-j)(2T+1-t)} = C^i_{jt}$$

for any $i = 1, \ldots, n$, $t = 1, \ldots, 2T$ *and* $j = 1, \ldots, 2d_i$.

Proof. Let us first consider $\psi^i_{(2d_i+1-j)(2T-l)}$. By definition (2) we have it equal

$$|f_i(1-j+lr_i) - f_i(-j+lr_i)|,$$

which by the symmetry of f_i equals

$$|f_i(j-lr_i) - f_i(j-1-lr_i)|.$$

Therefore,

$$\psi^i_{(2d_i+1-j)(2T-l)} = \psi^i_{jl}.$$

Now, let us consider

$$C^i_{(2d_i+1-j)(2T+1-t)} = \sum_{l=min\{2T+1-t, Z^i_{2d_i+1-j}\}}^{max\{2T+1-t, Z^i_{2d_i+1-j}\}-1} \psi^i_{(2d_i+1-j)l}.$$

By definitions of $Z^i_{2d_i+1-j}$ and Z^i_j, we have

$$Z^i_{2d_i+1-j} = 2T - Z^i_j + \sigma,$$

where $\sigma = 0$ if $(2j-1)/r_i$ is integer and $\sigma = 1$ otherwise. Thus,

$$C^i_{(2d_i+1-j)(2T+1-t)} = \sum_{l=2T+min\{1-t, -Z^i_j+\sigma\}}^{2T+max\{1-t, -Z^i_j+\sigma\}-1} \psi^i_{(2d_i+1-j)l},$$

or

$$C^i_{(2d_i+1-j)(2T+1-t)} = \sum_{l=2T-max\{t-1,Z^i_j-\sigma\}}^{2T-min\{t-1,Z^i_j-\sigma\}-1} \psi^i_{(2d_i+1-j)l}. \tag{4}$$

By substituting l by $2T - l$ on the right hand side of (4), we obtain

$$C^i_{(2d_i+1-j)(2T+1-t)} = \sum_{l=min\{t-1,Z^i_j-\sigma\}+1}^{max\{t-1,Z^i_j-\sigma\}} \psi^i_{(2d_i+1-j)(2T-l)}.$$

Since

$$\psi^i_{(2d_i+1-j)(2T-l)} = \psi^i_{jl},$$

we have

$$C^i_{(2d_i+1-j)(2T+1-t)} = \sum_{l=min\{t,Z^i_j-\sigma+1\}}^{max\{t,Z^i_j-\sigma+1\}-1} \psi^i_{jl}.$$

If $\sigma = 1$, then

$$C^i_{(2d_i+1-j)(2T+1-t)} = \sum_{l=min\{t,Z^i_j\}}^{max\{t,Z^i_j\}-1} \psi^i_{jl} = C^i_{jt},$$

and the lemma holds. Otherwise,

$$C^i_{(2d_i+1-j)(2T+1-t)} = \sum_{l=min\{t,Z^i_j+1\}}^{max\{t,Z^i_j+1\}-1} \psi^i_{jl}.$$

However, $\sigma = 0$ implies that $(2j-1)/r_i$ is integer. Consequently, by the symmetry of f_i, we have

$$\psi^i_{jZ^i_j} = 0,$$

and

$$C^i_{(2d_i+1-j)(2T+1-t)} = \sum_{l=min\{t,Z^i_j+1\}}^{max\{t,Z^i_j+1\}-1} \psi^i_{jl} = \sum_{l=min\{t,Z^i_j\}}^{max\{t,Z^i_j\}-1} \psi^i_{jl} = C^i_{jt},$$

which proves the lemma. □

The following Lemma 3 is a direct corollary of Lemma 2.

LEMMA 3. *We have*

$$C^i_{(2d_i+1-j)t} = C^i_{j(2T+1-t)}$$

for any $i = 1, \dots, n$, $t = 1, \dots, 2T$ *and* $j = 1, \dots, 2d_i$.

LEMMA 4. *We have*

$$C^i_{j(2T+1-t)} \leqslant C^i_{jt}$$

for any $i = 1, \dots, n$, $t = T+1, \dots, 2T$ *and* $j = 1, \dots, d_i$.

Proof. It is obvious that the lemma holds for $Z^i_j \leqslant 2T + 1 - t$. Otherwise $2T + 1 - t < Z^i_j$. Consider $t' = 2T + 1 - t$ and t. We have $t + t' > 2Z^i_j$ since $Z^i_j \leqslant T$. Thus, by Lemma 1, we obtain $C^i_{jt} \geqslant C^i_{jt'} = C^i_{j(2T+1-t)}$. □

LEMMA 5. *We have*

$$C^i_{(2d_i+1-j)t} \leqslant C^i_{jt}$$

for any $i = 1, \dots, n$, $t = 1, \dots, T$ *and* $j = d_i + 1, \dots, 2d_i$.

Proof. By Lemmas 4 and 2, respectively, we have

$$C^i_{(2d_i+1-j)t} \leqslant C^i_{(2d_i+1-j)(2T+1-t)} = C^i_{jt}.$$

□

4. Folding, Shuffle, and Unfolding of Even Sequences

Consider a feasible sequence $\alpha = \alpha_1, \dots, \alpha_T, \alpha_{T+1}, \dots, \alpha_{2T}$ for an even instance with demands $2d_1, \dots, 2d_n$. We show in this section how to construct a feasible sequence β for $d_1, \dots, d_n$ such that $F(\alpha) \geqslant F(\beta^2)$. The construction goes through three steps: folding, shuffle, and unfolding. The folding replaces α by a sequence of T ordered pairs $(\alpha_1, \alpha_{2T}), \dots, (\alpha_t, \alpha_{2T+1-t}), \dots, (\alpha_T, \alpha_{T+1})$. The shuffle shuffles copies inside of each pair producing a sequence $(\alpha'_1, \alpha'_{2T}), \dots, (\alpha'_t, \alpha'_{2T+1-t}), \dots, (\alpha'_T, \alpha'_{T+1})$, where $\{\alpha_t, \alpha_{2T+1-t}\} = \{\alpha'_t, \alpha'_{2T+1-t}\}$ for $t = 1, \dots, T$. Finally, the unfolding unfolds the outcome of the shuffle into a sequence $\alpha' = \alpha'_1, \dots, \alpha'_T, \alpha'_{T+1}, \dots, \alpha'_{2T}$ for $2d_1, \dots, 2d_n$. The shuffle uses the *Hall's theorem*, actually its special case that is the existence of a complete matching in a regular bipartite graph, to ensure that each product i occurs exactly d_i times in each of the two halves of α'. We show later that this three-step construction does not increase the cost of solution and consequently either half of α' can be taken as the required β. This proof is based on a crucial observation which is that the construction does not "push" the products further from their ideal positions than they were in the original sequence, and thus the assignment cost does not increase. We show all details of the construction in the next three subsections, where we denote S_α by S for simplicity.

4.1. FOLDING

Define

$$F(i,j,l) = \begin{cases} (i,j,l) & \text{if } j \leqslant d_i \text{ and } l \leqslant T, \\ (i, 2d_i+1-j, l) & \text{if } j > d_i \text{ and } l \leqslant T, \\ (i, j, 2T+1-l) & \text{if } j \leqslant d_i \text{ and } l > T, \\ (i, 2d_i+1-j, 2T+1-l) & \text{if } j > d_i \text{ and } l > T, \end{cases}$$

for $(i,j,l) \in S$. Let $F(S) = \{F(i,j,l) | (i,j,l) \in S\}$.

We observe the following.

LEMMA 6. *Let $t(i,j) = \{l | (i,j,l) \in F(S)\}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, d_i$. We have*

$$1 \leqslant |t(i,j)| \leqslant 2.$$

Proof. Consider (i,j,l) and $(i, 2d_i+1-j, l')$ from S for $i = 1, \ldots, n$ and $j = 1, \ldots, d_i$. We observe that $F(i,j,l)$ and $F(i, 2d_i+1-j, l')$ are the only two triples that can fall into $t(i,j)$. It is worth noticing that we may have $F(i,j,l) = F(i, 2d_i+1-j, l')$, which happens if $l = 2T+1-l'$. □

4.2. SHUFFLE

Define a bipartite graph $G = (V_1 \cup V_2, E)$ as follows:

$$V_1 = \{(i,j) | i = 1, \ldots, n; \; j = 1, \ldots, d_i\}, \quad V_2 = \{l | l = 1, \ldots, T\},$$

and

$$E = \{(i,j,l) | l \in t(i,j); \quad i = 1, \ldots, n; \; j = 1, \ldots, d_i\}.$$

Each node in graph $G = (V_1 \cup V_2, E)$ has degree either 1 or 2, see Lemma 6. Furthermore, if one end of an edge in E has degree 1, then the other has degree 1 as well. Let $G' = (V_1' \cup V_2', E')$ be a graph obtained by deleting all nodes of degree 1, along with connecting them edges, from G.

LEMMA 7. *Graph $G' = (V_1' \cup V_2', E')$ has two disjoint complete matchings M and M^c such that $M \cup M^c = E'$.*

Proof. Graph G' is a *regular* bipartite graph of degree 2, and, therefore, it has a complete matching M, see [12]. It can be observed that $M^c = E' - M$ is another complete matching in G', which proves the lemma. □

Define

$$L = M \cup (E - E')$$

and

$$R = M^c \cup (E - E').$$

We observe the following.

LEMMA 8. *For any* $i = 1, \dots, n$, *we have*

$$|\{l : (i, j, l) \in L \text{ for some } j\}| = |\{l : (i, j, l) \in R \text{ for some } j\}| = d_i.$$

Proof. The lemma immediately follows from the fact that there are exactly d_i nodes of the form (i, j), for some j, in V_1, and both L and R are complete matchings in G by Lemma 7. □

4.3. UNFOLDING

Define

$$U(i, j, l) = (i, j, l),$$

for $(i, j, l) \in L$, and

$$U(i, j, l) = (i, 2d_i + 1 - j, 2T + 1 - l),$$

for $(i, j, l) \in R$.

4.4. FOLDING, SHUFFLE AND UNFOLDING RESULT IN AN ASSIGNMENT

Let $FSU(S)$ be the set of triples obtained by the folding, shuffle, and unfolding triples from S. We shall prove that $FSU(S)$ satisfies (A) and (B). We first show that the composition of folding, shuffle and unfolding is a one-to-one mapping with respect to (i, j), Lemma 9, and with respect to l, Lemma 10.

LEMMA 9. *If* $(i, j', l'), (i, j'', l'') \in S$ *and* $j' \neq j''$, *then* $FSU(i, j', l') = (i, j_1, l_1)$, $FSU(i, j'', l'') = (i, j_2, l_2)$ *and* $j_1 \neq j_2$.

Proof. By contradiction. Suppose that $j_1 = j_2$. The FSU maps j' into either j' or $2d_i + 1 - j'$, and j'' into either j'' or $2d_i + 1 - j''$. Since $j' \neq j''$, then, without loss of generality, we have $j_1 = j'$ and $j_2 = 2d_i + 1 - j''$. Consequently, $j' = 2d_i + 1 - j''$. Thus, without loss of generality, $1 \leqslant j' \leqslant d_i$ and $d_i + 1 \leqslant j'' \leqslant 2d_i$. Therefore, the folding maps j' into j' and j'' into j'. Then, either $(i, j', t) \in M$ and $(i, j', s) \in M^c$ in the shuffle, for some $t \neq s$, or $(i, j', t) \in E - E'$ for some t. Consequently, either $FSU(i, j', l') = (i, j', s)$ and $FSU(i, j'', l'') = (i, j'', 2T + 1 - t)$, or $FSU(i, j', l') = (i, j', t)$ and $FSU(i, j'', l'') = (i, j'', 2T + 1 - t)$. Thus, $j_1 = j'$ and $j'' = j_2$ which results in a contradiction. □

LEMMA 10. *If* $(i', j', l'), (i'', j'', l'') \in S$ *and* $l' \neq l''$, *then* $FSU(i', j', l') = (i', j_1, l_1)$, $FSU(i'', j'', l'') = (i'', j_2, l_2)$ *and* $l_1 \neq l_2$.

Proof. By contradiction. Suppose that $l_1 = l_2$. The FSU maps l' into either l' or $2T + 1 - l'$, and l'' into either l'' or $2T + 1 - l''$. Since $l' \neq l''$, then, without loss of generality, we have $l_1 = l'$ and $l_2 = 2T + 1 - l''$. Consequently, $l' = 2T + 1 - l''$. Thus, without loss of generality, $1 \leqslant l' \leqslant T$ and $T + 1 \leqslant l'' \leqslant 2T$. Therefore, the folding maps l' into l' and l'' into l'. Then, either $(i', k', l') \in M$ and $(i'', k'', l') \in M^c$ or $(i', k', l') \in M^c$ and $(i'', k'', l') \in M$ in the shuffle. Consequently either $FSU(i', j', l') = (i', j_1, l')$ and $FSU(i'', j'', l'') = (i'', j_2, 2T + 1 - l')$ or $FSU(i', j', l') = (i', j_1, 2T + 1 - l')$ and $FSU(i'', j'', l'') = (i'', j_2, l')$. Thus, either $l_1 = l'$ and $l_2 = 2T + 1 - l'$ or $l_1 = 2T + 1 - l'$ and $l_2 = l'$ which results in a contradiction. □

We are now ready to show that.

LEMMA 11. $FSU(S)$ *satisfies both (A) and (B),*

Proof. Lemma 9 implies (B) for $FSU(S)$, and Lemma 10 implies (A) for $FSU(S)$. □

Furthermore, the composition of folding, shuffle and unfolding yields an assignment with its cost not exceeding the original one as we show in the following lemma.

LEMMA 12. *We have*

$$V(S) \geqslant V(FSU(S)).$$

Proof. Consider $(i, j, l) \in S$. If $1 \leqslant j \leqslant d_i$ and $1 \leqslant l \leqslant T$, then $FSU(i, j, l)$ is either (i, j, l) or $(i, 2d_i + 1 - j, 2T + 1 - l)$. Thus, $FSU(i, j, l)$ contributes either C^i_{jl} or $C^i_{(2d_i+1-j)(2T+1-l)}$ to $F(\mu)$, where $\mu = \alpha$ (FSU(S)). By Lemma 2, we have $C^i_{(2d_i+1-j)(2T+1-l)} = C^i_{jl}$, thus, $FSU(i, j, l)$ makes the same contribution to $F(\mu)$ as (i, j, l) does to $F(\alpha)$.

If $1 \leqslant j \leqslant d_i$ and $T+1 \leqslant l \leqslant 2T$, then $FSU(i,j,l)$ is either $(i,j,2T+1-l)$ or $(i,2d_i+1-j,l)$. Thus, $FSU(i,j,l)$ contributes either $C^i_{j(2T+1-l)}$ or $C^i_{(2d_i+1-j)l}$ to $F(\mu)$. By Lemma 4, we have $C^i_{j(2T+1-l)} \leqslant C^i_{jl}$. By Lemma 3, we have $C^i_{(2d_i+1-j)l} = C^i_{j(2T+1-l)}$, and, by Lemma 4, we get $C^i_{j(2T+1-l)} \leqslant C^i_{jl}$. Therefore, $C^i_{(2d_i+1-j)l} \leqslant C^i_{jl}$, and consequently the $FSU(i,j,l)$ contribution to $F(\mu)$ does not exceed the (i,j,l) contribution to $F(\alpha)$.

If $d_i+1 \leqslant j \leqslant 2d_i$ and $1 \leqslant l \leqslant T$, then $FSU(i,j,l)$ is either $(i,2d_i+1-j,l)$ or $(i,j,2T+1-l)$. Thus, $FSU(i,j,l)$ contributes either $C^i_{(2d_i+1-j)l}$ or $C^i_{j(2T+1-l)}$ to $F(\mu)$. By Lemma 5, we have $C^i_{(2d_i+1-j)l} \leqslant C^i_{jl}$. By Lemma 3, we have $C^i_{(2d_i+1-j)l} = C^i_{j(2T+1-l)}$, and, by Lemma 5, we get $C^i_{(2d_i+1-j)l} \leqslant C^i_{jl}$. Therefore, $C^i_{j(2T+1-l)} \leqslant C^i_{jl}$, and consequently the $FSU(i,j,l)$ contribution to $F(\mu)$ does not exceed the (i,j,l) contribution to $F(\alpha)$.

Finally, if $d_i+1 \leqslant j \leqslant 2d_i$ and $T+1 \leqslant l \leqslant 2T$, then $FSU((i,j,l))$ is either (i,j,l) or $(i,2d_i+1-j,2T+1-l)$. Thus, $FSU((i,j,l))$ contributes either C^i_{jl} or $C^i_{(2d_i+1-j)(2T+1-l)}$ to $F(\mu)$. By Lemma 2, we have $C^i_{(2d_i+1-j)(2T+1-l)} = C^i_{jl}$, thus, $FSU(i,j,l)$ makes the same contribution to $F(\mu)$ as (i,j,l) does to $F(\alpha)$. □

The folding and shuffle might cause $FSU(S)$ to fail (C). However, by Theorem 2 and Lemma 11 a feasible solution S' can be constructed for which

$$V(FSU(S)) \geqslant V(S'). \tag{5}$$

The main result of this section is as follows.

THEOREM 3. *Let*

$$\alpha = \alpha_1, \dots, \alpha_T, \alpha_{T+1}, \dots, \alpha_{2T},$$

be a feasible sequence for $2d_1, \dots, 2d_n$. *Then, a sequence*

$$\mu = \mu_1, \dots, \mu_T, \mu_{T+1}, \dots, \mu_{2T},$$

where i occurs d_i times in the first half $\mu_1, \dots, \mu_T$ and d_i times in the second half $\mu_{T+1}, \dots, \mu_{2T}$ can be constructed such that

$$F(\mu) \leqslant F(\alpha).$$

Proof. Consider sequence $\mu = \alpha(S')$. By Lemma 8 and Theorem 2 each i occurs d_i times in the first half of μ and d_i times in the second. Furthermore, by Theorem 1, Lemma 12 and (6), $F(\mu) \leqslant F(\alpha)$ as required. □

5. Optimality of Cyclic Solutions

We are now ready to prove the main result of this paper.

THEOREM 4. *Let β be an optimal sequence for $d_1, \dots, d_n$. Then β^m, $m \geqslant 1$, is optimal for $md_1, \dots, md_n$.*

Proof. By induction on m. The theorem obviously holds for $m = 1$. Suppose that the theorem holds for any $1 \leqslant m \leqslant k$. We prove that it also holds for $m = k+1$. Consider an optimal sequence $\alpha_1, \dots, \alpha_{mT}$ for $md_1, \dots, md_n$. If m is even, then by Theorem 3, this sequence can be transformed without cost increasing into a sequence $\mu_1, \dots, \mu_{mT/2}, \mu_{1+mT/2}, \dots, \mu_{mT}$, where i occurs $md_i/2$ times in each of the two halves of μ. Thus, each half must be optimal for $md_1/2, \dots, md_n/2$. Therefore, by the inductive assumption, each half is the concatenation of $m/2$ copies of β, and the theorem holds for even $m = k + 1$. If m is odd, then consider sequence $\beta\alpha$ for $(m + 1)d_1, \dots, (m + 1)d_n$. We have $F(\beta\alpha) = F(\beta) + F(\alpha)$. By Theorem 3, $\beta\alpha$ can be transformed without cost increasing into a sequence $\mu_1, \dots, \mu_{(m+1)T/2}, \mu_{1+(m+1)T/2}, \dots, \mu_{(m+1)T}$ where i occurs $(m + 1)d_i/2$ times in each of the two halves of μ. Thus, each half must be optimal for $(m + 1)d_1/2, \dots, (m + 1)d_n/2$. Therefore, by the inductive assumption, each half is the concatenation of $(m + 1)/2$ copies of β, and $F(\beta\alpha) = F(\beta) + F(\alpha) \geqslant (m + 1)F(\beta)$. Consequently, $F(\alpha) \geqslant mF(\beta)$ which proves the theorem for odd $m = k + 1$. This proves the theorem. □

It is worth observing that the constructive folding, shuffle, and unfolding operations (FSU) are used in this paper to prove the existence of optimal cyclic sequences rather than to actually construct optimal sequences. The latter can be obtained by first calculating the greatest common divisor m of $d_1, \dots, d_n$, then by using the algorithm given in [9, 10] to obtain an optimal sequence for $d_1/m, \dots, d_n/m$, and finally by concatenating the sequence m times to construct an optimal sequence for the original demands $d_1, \dots, d_n$.

6. Conclusions

We have proven that optimal JIT sequences are cyclic. This result provides an important theoretical support to the usual for just-in-time systems practice of repeating relatively short sequence to build a sequence for a longer time horizon, Monden [14] and Miltenburg [13]. It has also important consequences for the computational time complexity of all existing algorithms for PRV. All these time complexities depend on the magnitude of demands $d_1, \dots, d_n$ and consequently on the magnitude of number T. The only known polynomial time, with respect to T and n, optimization algorithm for JIT sequences has time complexity $O(T^3)$, see [9, 10]. Theorem 4 makes it possible to reduce each of these demands by the factor of m, where m is the greatest common divisor of numbers $d_1, \dots, d_n$, in the computations of optimal JIT sequences. The Euclid's algorithm can find m in $O(n \log T)$ steps, see for instance [5].

Furthermore, Theorem 4 is a step forward in tackling theoretically intriguing question of how succinct the encoding of optimal JIT sequence can be ? The answer to this question also pertains to the computational complexity of the PRV problem since the input of the problem can be made very short by encoding numbers $d_1, \ldots, d_n$ using $O(\sum_{i=1}^{n} \log d_i)$ bits. This encoding, however, makes all polynomial time, with respect to T and n, algorithms for the PRV problem pseudopolynomial time algorithms. Therefore, the question, see Kubiak [7], whether there is an algorithm with time complexity bounded by a polynomial function of $\log T$ and n remains open.

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